Relativistic motion of a charged particle in a plane electromagnetic wave with arbitrary amplitude

1976 J. Phys. A: Math. Gen. 9 1841

(http://iopscience.iop.org/0305-4470/9/11/008)

J. Phys. A: Math. Gen., Vol. 9, No. 11, 1976. Printed in Great Britain. © 1976

Relativistic motion of a charged particle in a plane electromagnetic wave with arbitrary amplitude

J Krüger and M Bovyn†

Seminarie voor Wiskundige Natuurkunde, University of Ghent, Krijgslaan 271-S9, B-9000 Gent, Belgium

Received 30 April 1976, in final form 6 July 1976

Abstract. The relativistic equations of motion for a charged particle in interaction with a plane electromagnetic wave with arbitrary amplitude are solved rigorously both in terms of the co-moving time and the inertial time. The special case of an harmonic wave is considered in detail and a discussion is given concerning the radiated energy and the radiation reaction.

1. Introduction

In the last decade the problem of the relativistic interaction of free electrons with plane electromagnetic waves has been extensively discussed (Sarachik and Schappert 1970 and references contained therein). The reason for this interest has been the development of high-power optical lasers which can produce large field intensities. Recently (Kupersztych 1976) the same problem was studied from the point of view of Lorentz transformations. In this paper we analyse the motion both in terms of the proper time and inertial time, generalizing previous work (Sanderson 1966). In § 2 the general solution for the motion of a free electron interacting with a plane electromagnetic wave with an arbitrary amplitude is obtained. In § 3 the example of the harmonic wave is worked out in detail. A short discussion of the problem of radiation reaction is given in § 4.

2. General solution with arbitrary wave amplitude

Let us first introduce a suitable system of coordinates. In formulating the equations of motion of the particle under study, the proper time τ will be used as a parameter. For reasons of simplicity, we choose the origin of the rectangular Cartesian frame of reference $Oxyz$ at the spatial position occupied by the particle at the instant $\tau = 0$. The moment at which the particle passes through O is taken as origin for the time-measuring device of the inertial observer at rest in $Oxyz$. Further, the positive axis Ox is chosen in the direction of propagation of the plane electromagnetic wave with which the particle

† Aspirant NFWO.

interacts. Finally, the positive axis Oy is taken in such a way that in the Coulomb gauge this wave can be represented by the vector potential

$$\boldsymbol{A} = A(x-ct)\mathbf{1}_y. \tag{1}$$

We can describe the electromagnetic field by means of

$$\begin{aligned} \boldsymbol{E} &= -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} = -\frac{1}{c}\frac{\partial A}{\partial t}\mathbf{1}_y, \\ \boldsymbol{B} &= \boldsymbol{\nabla}\times\boldsymbol{A} = \frac{\partial A}{\partial x}\mathbf{1}_z. \end{aligned} \tag{2}$$

The equations of motion for the charged particle with rest mass m and electric charge e in interaction with the plane wave are

$$m\ddot{x} = \frac{e}{c}\dot{y}\frac{\partial A}{\partial x}, \tag{3a}$$

$$m\ddot{y} = -\frac{e}{c}\left(\dot{x}\frac{\partial A}{\partial x} + \dot{t}\frac{\partial A}{\partial t}\right) = -\frac{e}{c}\dot{A}, \tag{3b}$$

$$m\ddot{z} = 0, \tag{3c}$$

$$mc\ddot{t} = -\frac{e}{c}\dot{y}\left(\frac{1}{c}\frac{\partial A}{\partial t}\right), \tag{3d}$$

where a dot means differentiation with respect to the proper time τ. The inertial time t and the proper time τ are related by

$$\begin{aligned} \mathrm{d}\tau &= \mathrm{d}t\left\{1 - \frac{1}{c^2}\left[\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right]\right\}^{1/2} \\ \tau_{t=0} &= 0. \end{aligned} \tag{4}$$

The complete formulation of the problem requires the specification of the initial velocity $\boldsymbol{v}(0)$ and initial position. In the above convention, we have at $\tau = 0$:

$$x_0 = y_0 = z_0 = t_0 = 0. \tag{5}$$

The potential function A appearing in (1) is only determined apart from an additive constant. Nothing prevents us from choosing this constant in such a way that $A(0) = 0$.

Let us define the momentum $\boldsymbol{\pi}$ of the system by

$$\boldsymbol{\pi} = m\dot{\boldsymbol{r}} + (e/c)\boldsymbol{A}.$$

From equations ($3b$, c) it follows that π_y and π_z are constants of the motion. Taking into account the initial conditions, this can be expressed by:

$$\dot{y} = \dot{y}_0 - \frac{eA}{mc}, \tag{6}$$

$$\dot{z} = \dot{z}_0. \tag{7}$$

Equation (6) enables us to rewrite equations ($3a$, d) as

$$\ddot{x}=-\frac{1}{2}\frac{\partial}{\partial x}\left(-\frac{eA}{mc}+\dot{y}_0\right)^2, \tag{8}$$

$$c\ddot{t}=\frac{1}{2}\left(\frac{1}{c}\frac{\partial}{\partial t}\right)^2\left(-\frac{eA}{mc}+\dot{y}_0\right)^2. \tag{9}$$

Combined with equation (1), we have $\ddot{x}-c\ddot{t}=0$ from which it follows that

$$x-ct=-ac\tau,$$

in which equation (5) was taken into account and where $a=\dot{t}_0-c^{-1}\dot{x}_0(>0)$. This result makes A purely a function of τ, and the equations (8)–(9) may be rewritten as:

$$\ddot{x}=\frac{1}{2ac}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(-\frac{eA}{mc}+\dot{y}_0\right)^2,$$

$$c\ddot{t}=\frac{1}{2ac}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(-\frac{eA}{mc}+\dot{y}_0\right)^2.$$

After one integration, we find:

$$\dot{x}=\dot{x}_0+\frac{1}{2ac}\left[\left(-\frac{eA}{mc}+\dot{y}_0\right)^2-\dot{y}_0^2\right], \tag{10}$$

$$c\dot{t}=c\dot{t}_0+\frac{1}{2ac}\left[\left(-\frac{eA}{mc}+\dot{y}_0\right)^2-\dot{y}_0^2\right]. \tag{11}$$

Equations (6), (7), (10) and (11) may be integrated once more if the function A is known. For a periodic A-function with period $2\pi/a\omega$, i.e.,

$$A=a_0+\sum_{n=1}^{\infty}(a_n\cos na\omega\tau+b_n\sin na\omega\tau),$$

where $a_0=-\Sigma_{n=1}^{\infty}a_n$ by virtue of $A(0)=0$, it is possible, at least in principle, to express x, y, z and τ by means of series expansions in terms of the inertial time t. Hereby, a new period $2\pi/\omega^*$ appears. Indeed, the integration of (11) yields

$$t=\frac{a\omega}{\omega^*}\tau+f(\tau),$$

where f is a periodic function with period $2\pi/a\omega$. Solving for τ gives

$$\tau=\frac{\omega^*}{a\omega}t+g(t)$$

where g is some periodic function with period $2\pi/\omega^*$. ω^* is given by

$$\omega^*=a\omega\left[\dot{t}_0-\frac{e\dot{y}_0}{amc^3}a_0+\frac{e^2}{2am^2c^4}\left(a_0^2+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)\right)\right]^{-1}$$

with

$$a_0=\frac{a\omega}{2\pi}\int_0^{2\pi/a\omega}A\,\mathrm{d}\tau,\qquad a_0^2+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)=\frac{a\omega}{2\pi}\int_0^{2\pi/a\omega}A^2\,\mathrm{d}\tau.$$

3. Example of the harmonic wave

As an example, we consider the case of A being simply a sinusoidal function:

$$A = -\frac{cB_0}{\omega}\sin\omega\left(t-\frac{x}{c}\right),$$

in which B_0 denotes the magnetic field strength in $x=0$ at the time $t=0$. The exact expressions for x, y, z and t are easy to calculate in terms of τ. We find:

$$x = \dot{x}_0\tau + \frac{c}{4a}\left(\frac{\Omega}{\omega}\right)^2\left(\tau - \frac{\sin 2a\omega\tau}{2a\omega}\right) + \frac{\dot{y}_0}{a}\left(\frac{\Omega}{\omega}\right)\frac{1-\cos a\omega\tau}{a\omega}, \tag{12a}$$

$$y = \dot{y}_0\tau + c\left(\frac{\Omega}{\omega}\right)\frac{1-\cos a\omega\tau}{a\omega}, \tag{12b}$$

$$z = \dot{z}_0\tau, \tag{12c}$$

$$t = \left(a+\frac{\dot{x}_0}{c}\right)\tau + \frac{1}{4a}\left(\frac{\Omega}{\omega}\right)^2\left(\tau - \frac{\sin 2a\omega\tau}{2a\omega}\right) + \frac{\dot{y}_0}{ac}\left(\frac{\Omega}{\omega}\right)\frac{1-\cos a\omega\tau}{a\omega}, \tag{12d}$$

where $\Omega = eB_0/mc$ denotes the cyclotron frequency corresponding to a uniform magnetic field of strength B_0. In the non-relativistic limit $a\to 1, \tau\to t, \Omega \ll \omega$, the solution reduces (in first order of $1/c$) to:

$$x \cong v_x(0)t + \frac{c}{4}\left(\frac{\Omega}{\omega}\right)^2\left(t - \frac{\sin 2\omega t}{2\omega}\right) + v_y(0)\frac{\Omega}{\omega}\,\frac{1-\cos\omega t}{\omega},$$

$$y \cong v_y(0)t + c\left(\frac{\Omega}{\omega}\right)\frac{1-\cos\omega t}{\omega},$$

$$z \cong v_z(0)t.$$

Sanderson (1966) has obtained a similar result but with different initial conditions ($\boldsymbol{v}(0)=0$).

In the relativistic case, the inversion of the formula (12d) gives rise to the new period $2\pi/\omega^*$ with

$$\omega^* = a\omega\left[\dot{t}_0 + \frac{1}{4a}\left(\frac{\Omega}{\omega}\right)^2\right]^{-1} = a\omega\left[a + \frac{\dot{x}_0}{c} + \frac{1}{4a}\left(\frac{\Omega}{\omega}\right)^2\right]^{-1}.$$

The coefficients in the Fourier series expansions of τ, x, y and z can easily be written as integrals over the period $2\pi/\omega^*$. In the special case $\dot{y}_0=0$, these integrals can be evaluated in terms of known functions. Indeed, the inversion of (12d) is then equivalent to the expansion of the excentric anomaly E in terms of the mean anomaly M of the Kepler problem. When $\dot{y}_0=0$, equation (12d) may be rewritten in the form

$$M = E - \epsilon\sin E,$$

where

$$M = 2\omega^* t, \qquad E = 2a\omega\tau, \qquad \epsilon = \frac{1}{4}\left(\frac{\Omega}{a\omega}\right)^2\frac{\omega^*}{\omega}.$$

Making use of a known result (Watson 1944), we obtain directly:

$$\tau = \frac{\omega^*}{a\omega}t + \frac{1}{2a\omega}\sum_{n=1}^{\infty}\frac{2}{n}J_n(n\epsilon)\sin 2n\omega^* t.$$

Proceeding in the same way, we find for x, y and z:

$$x=\left(1-\frac{\omega^*}{\omega}\right)ct-\frac{c}{2\omega}\sum_{n=1}^{\infty}\frac{2}{n}J_n(n\epsilon)\sin 2n\omega^* t,$$

$$y=\frac{c\Omega}{a\omega^2}\left(1-\sum_{n=0}^{\infty}\frac{1}{2n+1}\{J_n[(n+\tfrac{1}{2})\epsilon]-J_{n+1}[(n+\tfrac{1}{2})\epsilon]\}\cos(2n+1)\omega^* t\right)$$

$$z=\dot{z}_0\left(\frac{\omega^*}{a\omega}t+\frac{1}{2a\omega}\sum_{n=1}^{\infty}\frac{2}{n}J_n(n\epsilon)\sin 2n\omega^* t\right).$$

4. Radiated energy and radiation reaction

The instantaneous radiated energy of the particle is, using Larmor's formula (Jackson 1962):

$$P_{\text{rad}}=\frac{2e^2}{3c^3}(\ddot{r}^2-c\ddot{t}^2)=\frac{2e^4}{3m^2c^5}\left(\frac{\mathrm{d}A}{\mathrm{d}\tau}\right)^2\bigg|_{\tau=\tau(t)}. \tag{13}$$

For the example of § 3, equation (13) yields

$$P_{\text{rad}}=\frac{2e^2a^2\Omega^2}{3c}\left(\sum_{n=0}^{\infty}\frac{1}{2n+1}\{J_n[(n+\tfrac{1}{2})\epsilon]-J_{n+1}[(n+\tfrac{1}{2})\epsilon]\}\cos(2n+1)\omega^* t\right)^2. \tag{14}$$

The process can also be understood as the scattering of the incident wave by an electron: the coefficients of the Fourier series in equation (14) are proportional to the classical probability amplitude for the n-photon process. From the viewpoint of scattering the particle itself does not exhibit any essential radiation damping effect. This can easily be seen in the non-relativistic limit. The equations of motion including radiation damping can then be approximated by

$$m\ddot{\boldsymbol{r}}-mT\dddot{\boldsymbol{r}}=\boldsymbol{F}_{\text{ext}}, \tag{15}$$

where $T=2e^2/3mc^3$ and where $\Omega\ll\omega\ll T^{-1}$. In this approximation $\boldsymbol{F}_{\text{ext}}$ can be considered as independent of $\boldsymbol{r}$. Then, the physically acceptable solution of equation (15) reads:

$$\ddot{\boldsymbol{r}}=\frac{\mathrm{e}^{t/T}}{mT}\int_t^{\infty}\mathrm{e}^{-t'/T}\boldsymbol{F}_{\text{ext}}(t')\,\mathrm{d}t'. \tag{16}$$

Equation (16) gives no damping for a periodic external force. If the incoming wave is exponentially damped, equation (16) does not lead to a supplementary damping but the damping coefficient is the same as that of the incident wave. To first order of T, the motion of the particle is only affected by phase shifts, if the incoming wave is undamped. This means that the energy of the outgoing wave is totally extracted from the incident wave in the limits considered; the particle causes the scattering of the radiation without being damped itself. This apparent paradox can be explained by making the energy balance. Carrying out scalar multiplication by $\dot{\boldsymbol{r}}$ on both sides of (15) and integrating between two arbitrary moments t_1 and t_2 gives:

$$\tfrac{1}{2}[m\dot{\boldsymbol{r}}^2]_{t_1}^{t_2}+mT\int_{t_1}^{t_2}\ddot{\boldsymbol{r}}^2\,\mathrm{d}t=mT[\dot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}}]_{t_1}^{t_2}+\int_{t_1}^{t_2}\dot{\boldsymbol{r}}\cdot\boldsymbol{F}_{\text{ext}}\,\mathrm{d}t. \tag{17}$$

The first term on the right-hand side can be set equal to zero by proper choice of t_1 and t_2. In the absence of the external force equation (17) then necessarily leads to the decrease of the kinetic energy. However, in the presence of the external force, the kinetic energy supplemented by the radiation energy is fully compensated by the potential energy to first order in ωT, as can easily be checked on the example of § 3. However, for high intensity laser radiation Ω/ω can become considerably larger than unity. In this case the radiation reaction force can only be neglected if

$$\Omega^2 T \ll \omega. \tag{18}$$

Even if this effect is small, it may, over sufficiently long time, induce an alteration to the electrons' motion (Sanderson 1965).

5. Conclusion

The problem of the motion of a charged particle in a plane electromagnetic wave with an arbitrary amplitude is solved in terms of the proper time and the inertial time of the particle. The frequency of the motion $2\pi/\omega^*$ is related to the frequency $2\pi/\omega$ of the incoming wave by a Doppler shift. The special case for an incoming harmonic plane wave is considered in detail. For simple initial conditions (i.e. $\dot{y}(0) = 0$) the solution is explicitly obtained in the form of a series expansion of harmonics with frequency $n\omega^*$ ($n = 1, 2, 3, \ldots$). The problem of radiation reaction is briefly discussed: since the process can be understood as the scattering of the incident wave by the charged particle there is no essential damping effect to be observed in the non-relativistic limit.

Acknowledgment

It is a pleasure to thank Professor Dr C C Grosjean for constant encouragement and many helpful suggestions with respect to the final version of the manuscript.

References

Jackson J D 1962 *Classical Electrodynamics* (New York: Wiley) p 469
Kupersztych J 1976 *Nuovo Cim.* B **31** 1–11
Sanderson J J 1965 *Phys. Lett.* **18** 114–5
—— 1966 *Int. J. Electron.* **20** 87–9
Sarachik E S and Schappert S T 1970 *Phys. Rev.* D **1** 2738–53
Watson G N 1944 *A Treatise on the Theory of Bessel Functions* (Cambridge: Cambridge University Press) p 553